2014

(5th Semester)

MATHEMATICS

Paper No. : MATH–352

(Real Analysis)

Full Marks : 75

Time : 3 hours

(PART : B—DESCRIPTIVE)

(*Marks* : 50)

The figures in the margin indicate full marks for the questions

Answer **five** questions, taking **one** from each Unit

UNIT—I

1. State and prove Cantor's intersection theorem. 2+8=10

2. *(a)* Prove that every open cover of a set admits of a countable sub-cover. 4

(b) Prove that every open cover of a compact set admits of a finite sub-cover. 6

UNIT—II

3. *(a)* Evaluate the limit for the function

$$f(x) = \frac{x^2 - y^2}{x^2 + y^2}$$

when $(x, y) \to (0, 0)$. 4

(b) Prove that the range of a function continuous on a compact set is compact. 6

4. *(a)* Let $\lim\limits_{x \to a} f(x) = b$ and

let $b = (b_1, \ldots, b_m)$, $f = (f_1, \ldots, f_m)$.

Show that $\lim\limits_{x \to a} f_i(x) = b_i$, $1 \le i \le m$ and conversely. 6

(b) Show that the function

$$f(x, y) = x^2 + 2xy; \quad (x, y) \neq (1, 2)$$
$$f(x, y) = 0 \quad ; \quad (x, y) = (1, 2)$$

has a removable discontinuity at $(1, 2)$. 4

UNIT—III

5. *(a)* Let

$$f(x, y) = \begin{cases} \dfrac{xy}{\sqrt{x^2 + y^2}}; & \text{when } (x, y) \neq (0, 0) \\ 0 \quad ; & (x, y) = (0, 0) \end{cases}$$

Show that f is continuous and possesses partial derivatives but not differentiable at $(0, 0)$. 5

(b) If

$$u = \frac{x+y}{1-xy}, \quad v = \frac{(x+y)(1-xy)}{(1+x^2)(1+y^2)}$$

find $\frac{\partial(u, v)}{\partial(x, y)}$. 5

6. (a) Prove that a function which is differentiable at a point admits of partial derivatives at the point. 6

(b) Let

$$f(x, y) = \begin{cases} \frac{x^2 y}{x^4 + y^2}; & x^2 + y^2 \neq (0, 0) \\ 0 & ; x^2 + y^2 = (0, 0) \end{cases}$$

Show that f is a directional derivative at $(0, 0)$ in any arbitrary direction $\beta = (l, m)$, $l^2 + m^2 = 1$, but f is not continuous at $(0, 0)$. 4

UNIT—IV

7. State and prove Taylor's theorem. 2+8=10

8. (a) If

$$f(x, y) = \begin{cases} (x^2 + y^2) \tan^{-1}(y/x); & x \neq 0 \\ \frac{\pi}{2} y^2 & ; x = 0 \end{cases}$$

show that $f_{xy}(0, 0) \neq f_{yx}(0, 0)$. 5

(b) Find all the maxima and minima of the function given by

$$f(x, y) = x^3 + y^3 - 63(x+y) + 12xy$$ 5

UNIT—V

9. (a) Prove that every compact subset A of a metric space (X, d) is bounded. 5

(b) Let l_∞ be the set of all bounded numerical sequences $\{x_n\}$ in which the metric d is defined by

$$d(x, y) = \sup_n |x_n - y_n| \;\forall\; x = \{x_n\},\; y = \{y_n\} \in l_\infty$$

Show that (l_∞, d) is complete space. 5

10. Show that the set R^n of all ordered n-tuples with the function d defined by

$$d(x, y) = \left(\sum_{i=1}^{n} (x_i - y_i)^2 \right)^{1/2}$$

$$\forall x = (x_1, x_2, ..., x_n)$$

$$y = (y_1, y_2, ..., y_n) \in R^n$$

is a metric space. 10

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2014

(5th Semester)

MATHEMATICS

Paper No. : MATH–352

(Real Analysis)

(PART : A—OBJECTIVE)

(*Marks* : 25)

Answer **all** questions

SECTION—A

(Multiple Choice)

(*Marks* : 10)

Each question carries 1 mark

Put a Tick ☑ mark against the correct answer in the box provided :

1. A non-void set is said to be closed if every limit point thereof belongs to the set. A void set is called

(*a*) open ☐

(*b*) closed ☐

(*c*) bounded ☐

(*d*) interior point ☐

2. A set is said to be compact if it is both

(a) bounded and closed ☐

(b) open and closed ☐

(c) bounded and open ☐

(d) None of the above ☐

3. If a function is derivable at a point of its domain, then the function is

(a) continuous ☐

(b) uniformly continuous ☐

(c) not continuous ☐

(d) None of the above ☐

4. If f is a continuous function defined on a compact set, then the image of f is

(a) complete ☐

(b) compact ☐

(c) countable ☐

(d) uncountable ☐

5. If $u = f(x, y)$ and $v = g(x, y)$ have continuous partial derivatives in a region R of the xy-plane, a necessary and sufficient condition that they satisfy a functional region $F(u, v) = 0$ is the Jacobian

(a) $\frac{\partial(u, v)}{\partial(x, y)} = F(u, v)$ ☐

(b) $\frac{\partial(x, y)}{\partial(u, v)} = 0$ ☐

(c) $\frac{\partial(u, v)}{\partial(x, y)} \neq 0$ ☐

(d) $\frac{\partial(u, v)}{\partial(x, y)} = 0$ ☐

6. The directional derivative of $f(x, y) = 2x^2 - xy + 5$ at $(1, 1)$ in the direction of a unit vector $\beta = \frac{1}{5}(3, -4)$ is

(a) $\frac{5}{13}$ ☐

(b) $\frac{13}{5}$ ☐

(c) $\frac{13}{3}$ ☐

(d) $\frac{3}{13}$ ☐

7. If (a, b) be a point of the domain contained in R^2 of a function f such that f_x and f_y are both differentiable at (a, b), then $f_{xy}(a, b) = f_{yx}(a, b)$. This theorem is named as

(a) Taylor's theorem ☐

(b) Young's theorem ☐

(c) Schwarz's theorem ☐

(d) None of the above ☐

8. When $AC - B^2 < 0$, $f(a, b)$ is neither $A(\neq 0)$ nor $B(\neq 0)$, then f is

(a) minimum at (a, b) ☐

(b) maximum or minimum at (a, b) ☐

(c) maximum at (a, b) ☐

(d) None of the above ☐

9. Every finite subset of a metric space is

(a) closed ☐

(b) open ☐

(c) open and closed ☐

(d) bounded ☐

10. If every Cauchy sequence of X converges to a point of X, then a metric space (X, d) is

(a) compact ☐

(b) interior ☐

(c) complete ☐

(d) closure ☐

SECTION—B

(Very Short Answer)

(*Marks* : 15)

Each question carries 3 marks

1. Show that the union of an arbitrary family of open sets is open.

2. State intermediate value theorem.

3. If $u_1 = \dfrac{x_2 x_3}{x_1}$, $u_2 = \dfrac{x_1 x_3}{x_2}$ and $u_3 = \dfrac{x_1 x_2}{x_3}$, prove that $J(u_1, u_2, u_3) = 4$.

4. State the necessary condition for an extreme value.

5. Let X be an infinite set with the discrete metric. Show that (X, d) is not compact.

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