2015

(5th Semester)

MATHEMATICS

Paper : MATH–352

(Real Analysis)

Full Marks : 75

Time : 3 hours

(PART : B—DESCRIPTIVE)

(*Marks* : 50)

The figures in the margin indicate full marks for the questions

Answer **five** questions, taking **one** from each Unit

UNIT—I

1. Define a limit point of a set. Prove that every infinite and bounded set has at least one limit point. 2+8=10

2. *(a)* Prove that every open cover of a compact set admits of a finite subcover. 6

(b) Show that a set is closed if and only if its complement is open. 4

UNIT—II

3. *(a)* Prove that a function continuous on a compact domain is uniformly continuous. 6

(b) Let $f : R^2 \to R$ be a function defined by

$$f(x, y) = \begin{cases} \dfrac{x^2 y}{x^4 + y^2}, & (x, y) \neq (0, 0) \\ 0 \quad , & (x, y) = (0, 0) \end{cases}$$

Then test the continuity of f at $(0, 0)$. 4

4. *(a)* State and prove intermediate value theorem. 1+5=6

(b) Let

$$\lim_{x \to a} f(x) = b \text{ and } b = (b_1, b_2, \cdots, b_m)$$
$$f = (f_1, f_2, \cdots, f_m)$$

Then show that

$$\lim_{x \to a} f_i(x) = b_i, \; 1 \leq i \leq m$$ 4

UNIT—III

5. *(a)* If α, β, γ are the roots of the equation in t, such that

$$\frac{u}{a+t} + \frac{v}{b+t} + \frac{w}{c+t} = 1$$

then prove that

$$\frac{\partial(u, v, w)}{\partial(\alpha, \beta, \gamma)} = -\frac{(\beta - \gamma)(\gamma - \alpha)(\alpha - \beta)}{(b-c)(c-a)(a-b)}$$ 6

(b) Prove that a function which is differentiable at a point admits of partial derivatives at the point. 4

6. If

$$f(x, y) = \begin{cases} \dfrac{x^3}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0 , & (x, y) = (0, 0) \end{cases}$$

show that—

(i) f is continuous at (0, 0);

(ii) directional derivative of f exists at (0, 0) in every direction;

(iii) f is not differentiable at (0, 0).

3+2+5=10

UNIT—IV

7. State and prove Young's theorem. 2+8=10

8. (a) Show that the function

$$f(x, y) = \begin{cases} \dfrac{xy(x^2 - y^2)}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0 , & (x, y) = (0, 0) \end{cases}$$

does not satisfy the conditions of Schwarz's theorem and

$$f_{xy}(0, 0) \neq f_{yx}(0, 0)$$

7

(b) Show that

$$f(x, y) = (y-x)^4 + (x-2)^4$$

has a minimum at (2, 2). 3

UNIT—V

9. (a) Let (X, d) be any metric space. Then show that the function d_1 defined by

$$d_1(x, y) = \frac{d(x, y)}{1+d(x, y)}, \quad \forall x, y \in X$$

is a metric on X. 6

(b) Let (X, d) be a metric space and let x, y, z be any three points of X, then show that

$$d(x, y) \geq |d(x, z) - d(z, y)|$$ 4

10. (a) Prove that every compact subset F of a metric space (X, d) is closed. 6

(b) In a metric space (X, d), prove that the intersection of an arbitrary family of closed sets is closed. 4

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2015

(5th Semester)

MATHEMATICS

Paper : MATH–352

(Real Analysis)

(PART : A—OBJECTIVE)

(*Marks* : 25)

Answer **all** questions

SECTION—A

(*Marks* : 10)

Each question carries 1 mark

Put a Tick ☑ mark against the correct answer in the box provided :

1. In R^2, the limit point of the set

$$\left\{\left(\frac{1}{m}, \frac{1}{n}\right); m \in N, n \in N\right\}$$

is

(a) (0, 0) ☐

(b) (1, 1) ☐

(c) (0, 1) ☐

(d) None of the above ☐

2. A set is said to be compact if and only if it is

(a) bounded ☐

(b) both bounded and closed ☐

(c) open ☐

(d) None of the above ☐

3. If every open cover of the set admits a finite subcover, it is said to have

(a) the Cantor intersection property ☐

(b) the Lindeloff covering property ☐

(c) the Heine-Borel property ☐

(d) None of the above ☐

4. If $x+y+z=u$, $y+z=uv$, $z=uvw$, then

$$\frac{\partial(x, y, z)}{\partial(u, v, w)}$$

is equal to

(a) u^2v ☐

(b) uv^2 ☐

(c) uv ☐

(d) u^2v^2 ☐

5. Let f be a real valued function with an open set $D \subset R^n$ as its domain. Then the function admits of directional derivative at every point where it admits of

(a) continuous first-order partial derivatives ☐

(b) first-order partial derivatives ☐

(c) second-order partial derivatives ☐

(d) None of the above ☐

6. Let X be a non-empty set and d is a function from $X \times X$ into R such that $d(x, y) = 0$ if and only if $x = y$. Then (X, d) is a metric space if $\forall\, x, y, z \in X$

(a) $d(x, y) = d(y, x)$ ☐

(b) $d(x, y) \le d(x, z) + d(y, z)$ ☐

(c) $d(x, z) + d(z, y) \le d(x, y)$ ☐

(d) None of the above ☐

7. If

$$f(x, y) = \begin{cases} \dfrac{xy^2}{x^2 + y^4}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

then the directional derivative of f at $(0, 0)$

(a) does not exist ☐

(b) exists in all directions ☐

(c) exists in a particular direction ☐

(d) None of the above ☐

8. If (a, b) be a point of the domain $D \subset R^2$ of a real valued function f such that f_x exists in a certain neighbourhood of (a, b) and f_{xy} is continuous at (a, b), then

(a) $f_{yx}(a, b) = f_{xy}(a, b)$ ☐

(b) $f_{yx}(a, b) > f_{xy}(a, b)$ ☐

(c) $f_{yx}(a, b) < f_{xy}(a, b)$ ☐

(d) None of the above ☐

9. A metric space (X, d) is said to be complete if

(a) every Cauchy sequence in X diverges to a point of X ☐

(b) every Cauchy sequence in X converges to a point of X ☐

(c) there exists no Cauchy sequence in X ☐

(d) None of the above ☐

10. If $f(x, y) = x^3 + y^3 - 3x - 12y + 20$, then f is

(a) maximum at (1, 2) ☐

(b) minimum at (−1, −2) ☐

(c) minimum at (1, 2) ☐

(d) None of the above ☐

SECTION—B

(*Marks* : 15)

Each question carries 3 marks

Answer the following :

1. Show that $f(x, y) = y^2 + x^2y + x^4$ has a minimum at (0, 0).

2. Show that the intersection of any finite family of open sets is open.

3. Define closure of a set. Show that a set A is closed if and only if $\overline{A} = A$.

4. Show that the function

$$f(x, y) = \begin{cases} \dfrac{xy}{\sqrt{x^2 + y^2}}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

is not differentiable at (0, 0).

5. Prove that every closed subset of a compact metric space is compact.

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